On the 8-Square Imaginaries.

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I write throughout 0 to denote positive unity, and uniting with it the seven imaginaries $1, \ldots 7$, form an octavic system $0, 1, 2, 3, 4, 5, 6, 7$, the laws of combination being

$$0^2 = 0,\ 1^2 = 2^2 = 3^2 = 4^2 = 5^2 = 6^2 = 7^2 = -0,$$

$$\begin{array}{lll} 123 = \varepsilon_1, & 145 = \varepsilon_2, & 167 = \varepsilon_3, \\ & 246 = \varepsilon_4, & 257 = \varepsilon_5, \\ & 347 = \varepsilon_6, & 356 = \varepsilon_7, \end{array}$$

where $\varepsilon = \pm$, viz. each ε has a determinate value $+$ or $-$ as the case may be; and where the formula, $123 = \varepsilon_1$, denotes the six equations

$$\begin{array}{lll} 23 = \ \ \varepsilon_1 1, & 31 = \ \ \varepsilon_1 2, & 12 = \ \ \varepsilon_1 3, \\ 32 = -\varepsilon_1 1, & 13 = -\varepsilon_1 2, & 21 = -\varepsilon_1 3, \end{array}$$

and so for the other formulæ: the multiplication table of the eight symbols thus is

	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	-0	$\varepsilon_1 3$	$-\varepsilon_1 2$	$\varepsilon_2 5$	$-\varepsilon_2 4$	$\varepsilon_3 7$	$-\varepsilon_3 6$
2	2	$-\varepsilon_1 3$	-0	$\varepsilon_1 1$	$\varepsilon_4 6$	$\varepsilon_5 7$	$-\varepsilon_4 4$	$-\varepsilon_5 5$
3	3	$\varepsilon_1 2$	$-\varepsilon_1 1$	-0	$\varepsilon_6 7$	$\varepsilon_7 6$	$-\varepsilon_7 5$	$-\varepsilon_6 4$
4	4	$-\varepsilon_2 5$	$-\varepsilon_4 6$	$-\varepsilon_6 7$	-0	$\varepsilon_2 1$	$\varepsilon_4 2$	$\varepsilon_6 3$
5	5	$\varepsilon_2 4$	$-\varepsilon_5 7$	$-\varepsilon_7 6$	$-\varepsilon_2 1$	-0	$\varepsilon_7 3$	$\varepsilon_5 2$
6	6	$-\varepsilon_3 7$	$\varepsilon_4 4$	$\varepsilon_7 5$	$-\varepsilon_4 2$	$-\varepsilon_7 3$	-0	$\varepsilon_3 1$
7	7	$\varepsilon_3 6$	$\varepsilon_5 5$	$\varepsilon_6 4$	$-\varepsilon_6 3$	$-\varepsilon_5 2$	$-\varepsilon_3 1$	-0

Hence if 0, 1, 2, 3, 4, 5, 6, 7 and 0′, 1′, 2′, 3′, 4′, 5′, 6′, 7′ denote ordinary algebraical magnitudes, and we form the product

$$(0\mathbf{0}+1\mathbf{1}+2\mathbf{2}+3\mathbf{3}+4\mathbf{4}+5\mathbf{5}+6\mathbf{6}+7\mathbf{7})(0'\mathbf{0}+1'\mathbf{1}+2'\mathbf{2}+3'\mathbf{3}+4'\mathbf{4}+5'\mathbf{5}+6'\mathbf{6}+7'\mathbf{7}),$$

this is at once found to be =

$$\begin{array}{rll}
 & (00'-11'-22'-33'-44'-55'-66'-77') & \mathbf{0} \\
+ & (01'+0'1+\varepsilon_1 23+\varepsilon_2 45+\varepsilon_3 67) & \mathbf{1} \\
+ & (02'+0'2+\varepsilon_1 31+\varepsilon_4 46+\varepsilon_5 57) & \mathbf{2} \\
+ & (03'+0'3+\varepsilon_1 12+\varepsilon_6 47+\varepsilon_7 56) & \mathbf{3} \\
+ & (04'+0'4+\varepsilon_1 51+\varepsilon_4 62+\varepsilon_6 73) & \mathbf{4} \\
+ & (05'+0'5+\varepsilon_2 14+\varepsilon_5 72+\varepsilon_7 63) & \mathbf{5} \\
+ & (06'+0'6+\varepsilon_3 71+\varepsilon_4 24+\varepsilon_7 35) & \mathbf{6} \\
+ & (07'+0'7+\varepsilon_3 16+\varepsilon_5 25+\varepsilon_6 34) & \mathbf{7}
\end{array}$$

where 12 is written to denote $12'-1'2$, and so in other cases.

The sum of the squares of the eight coefficients of $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$, $\mathbf{3}$, $\mathbf{4}$, $\mathbf{5}$, $\mathbf{6}$, $\mathbf{7}$ respectively, will, if certain terms destroy each other, be

$$=(0^2+1^2+2^2+3^2+4^2+5^2+6^2+7^2)\,(0'^2+1'^2+2'^2+3'^2+4'^2+5'^2+6'^2+7'^2)$$

viz., the sum of the squares contains the several terms

$$\begin{array}{ccccccc}
\varepsilon_1\varepsilon_2 23.45\,, & \varepsilon_1\varepsilon_3 23.67\,, & \varepsilon_1\varepsilon_4 31.46\,, & \varepsilon_1\varepsilon_5 31.57\,, & \varepsilon_1\varepsilon_6 12.47\,, & \varepsilon_1\varepsilon_7 12.56\,, & \varepsilon_2\varepsilon_3 45.67\,, \\
\varepsilon_4\varepsilon_7 24.35\,, & \varepsilon_4\varepsilon_6 62.73\,, & \varepsilon_2\varepsilon_7 14.63\,, & \varepsilon_2\varepsilon_6 51.73\,, & \varepsilon_2\varepsilon_5 14.72\,, & \varepsilon_2\varepsilon_4 51.62\,, & \varepsilon_4\varepsilon_5 46.57\,, \\
\varepsilon_5\varepsilon_6 25.34\,, & \varepsilon_5\varepsilon_7 72.63\,, & \varepsilon_3\varepsilon_6 16.34\,, & \varepsilon_3\varepsilon_7 71.35\,, & \varepsilon_3\varepsilon_4 71.24\,, & \varepsilon_3\varepsilon_5 16.25\,, & \varepsilon_6\varepsilon_7 47.56\,,
\end{array}$$

and observing that $\mathbf{21}=-\mathbf{12}$ etc., and that we have identically $\mathbf{23}\cdot\mathbf{45}+\mathbf{24}\cdot\mathbf{53}+\mathbf{25}\cdot\mathbf{34}=$ zero, etc., then the three terms of each column will vanish, provided a proper relation exists between the ε's, viz., the conditions which we thus obtain are

$$\begin{array}{lllll}
\varepsilon_1\varepsilon_2 & = & -\varepsilon_4\varepsilon_7 & = & \varepsilon_5\varepsilon_6\,, \\
\varepsilon_1\varepsilon_3 & = & -\varepsilon_4\varepsilon_6 & = & \varepsilon_5\varepsilon_7\,, \\
\varepsilon_1\varepsilon_4 & = & -\varepsilon_3\varepsilon_6 & = & -\varepsilon_2\varepsilon_7\,, \\
\varepsilon_1\varepsilon_5 & = & \varepsilon_3\varepsilon_7 & = & \varepsilon_2\varepsilon_6\,, \\
\varepsilon_1\varepsilon_6 & = & \varepsilon_2\varepsilon_5 & = & -\varepsilon_3\varepsilon_4\,, \\
\varepsilon_1\varepsilon_7 & = & -\varepsilon_2\varepsilon_4 & = & \varepsilon_3\varepsilon_5\,, \\
\varepsilon_2\varepsilon_3 & = & -\varepsilon_4\varepsilon_5 & = & \varepsilon_6\varepsilon_7\,.
\end{array}$$

We may without loss of generality assume $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +$; the equations then become

$$\begin{array}{rcrcl} + & = & -\varepsilon_4\varepsilon_7 & = & \varepsilon_5\varepsilon_6 \\ + & = & -\varepsilon_4\varepsilon_6 & = & \varepsilon_5\varepsilon_7 \\ + & = & -\varepsilon_4\varepsilon_5 & = & \varepsilon_6\varepsilon_7 \\ \varepsilon_4 & = & -\varepsilon_6 & = & -\varepsilon_7 \\ \varepsilon_5 & = & \varepsilon_7 & = & \varepsilon_6 \,, \\ \varepsilon_6 & = & \varepsilon_5 & = & -\varepsilon_4 \\ \varepsilon_7 & = & -\varepsilon_4 & = & \varepsilon_5 \end{array}$$

and writing $\theta = \pm$ at pleasure, these are all satisfied if $-\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \theta$. The terms written down all disappear, and the sum of the squares of the eight coefficients thus becomes equal to the product of two sums each of them of eight squares, viz., this is the case if $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +$, $-\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \theta$, θ being $= \pm$ at pleasure: the resulting system of imaginaries may be said to be an 8-square system.

We may inquire whether the system is associative; for this purpose, supposing in the first instance that the ε's remain arbitrary, we form the complete system of the values of the triplets $12 \cdot 3$, $1 \cdot 23$, etc. (read the top line $12 \cdot 3 = -\varepsilon_1 0$, $1 \cdot 23 = -\varepsilon_1 0$, the next line $12 \cdot 4 = \varepsilon_1\varepsilon_6 7$, $1 \cdot 24 = \varepsilon_3\varepsilon_4 7$, and so in other cases):

$12 \cdot 3 =$	$1 \cdot 23 =$	$-\varepsilon_1$	$-\varepsilon_1$	0
$12 \cdot 4 =$	$1 \cdot 24 =$	$\varepsilon_1\varepsilon_6$	$\varepsilon_3\varepsilon_4$	7
$12 \cdot 5 =$	$1 \cdot 25 =$	$\varepsilon_1\varepsilon_7$	$-\varepsilon_3\varepsilon_5$	6
$12 \cdot 6 =$	$1 \cdot 26 =$	$-\varepsilon_1\varepsilon_7$	$-\varepsilon_2\varepsilon_4$	5
$12 \cdot 7 =$	$1 \cdot 27 =$	$-\varepsilon_1\varepsilon_6$	$\varepsilon_2\varepsilon_5$	4
$13 \cdot 4 =$	$1 \cdot 34 =$	$-\varepsilon_1\varepsilon_4$	$-\varepsilon_3\varepsilon_6$	6
$13 \cdot 5 =$	$1 \cdot 35 =$	$-\varepsilon_1\varepsilon_5$	$\varepsilon_3\varepsilon_7$	7
$13 \cdot 6 =$	$1 \cdot 36 =$	$\varepsilon_1\varepsilon_4$	$\varepsilon_2\varepsilon_7$	4
$13 \cdot 7 =$	$1 \cdot 37 =$	$\varepsilon_1\varepsilon_5$	$-\varepsilon_2\varepsilon_6$	5
$14 \cdot 5 =$	$1 \cdot 45 =$	$-\varepsilon_2$	$-\varepsilon_2$	0
$14 \cdot 6 =$	$1 \cdot 46 =$	$\varepsilon_2\varepsilon_7$	$\varepsilon_1\varepsilon_4$	3
$14 \cdot 7 =$	$1 \cdot 47 =$	$\varepsilon_2\varepsilon_5$	$-\varepsilon_1\varepsilon_6$	2
$15 \cdot 6 =$	$1 \cdot 56 =$	$-\varepsilon_2\varepsilon_4$	$-\varepsilon_1\varepsilon_7$	2
$15 \cdot 7 =$	$1 \cdot 57 =$	$-\varepsilon_2\varepsilon_6$	$\varepsilon_1\varepsilon_5$	3
$16 \cdot 7 =$	$1 \cdot 67 =$	$-\varepsilon_3$	$-\varepsilon_3$	0
$23 \cdot 4 =$	$2 \cdot 34 =$	$\varepsilon_1\varepsilon_2$	$-\varepsilon_5\varepsilon_6$	5
$23 \cdot 5 =$	$2 \cdot 35 =$	$-\varepsilon_1\varepsilon_2$	$-\varepsilon_4\varepsilon_7$	4
$23 \cdot 6 =$	$2 \cdot 36 =$	$\varepsilon_1\varepsilon_3$	$-\varepsilon_5\varepsilon_7$	7

$23 \cdot 7 =$	$2 \cdot 37 =$	$-\varepsilon_1\varepsilon_3$	$-\varepsilon_4\varepsilon_6$	6
$24 \cdot 5 =$	$2 \cdot 45 =$	$-\varepsilon_4\varepsilon_7$	$-\varepsilon_1\varepsilon_2$	3
$24 \cdot 6 =$	$2 \cdot 46 =$	$-\varepsilon_4$	$-\varepsilon_4$	0
$24 \cdot 7 =$	$2 \cdot 47 =$	$\varepsilon_3\varepsilon_4$	$\varepsilon_1\varepsilon_6$	1
$25 \cdot 6 =$	$2 \cdot 56 =$	$-\varepsilon_3\varepsilon_5$	$\varepsilon_1\varepsilon_7$	1
$25 \cdot 7 =$	$2 \cdot 57 =$	$-\varepsilon_5$	$-\varepsilon_5$	0
$26 \cdot 7 =$	$2 \cdot 67 =$	$-\varepsilon_4\varepsilon_6$	$-\varepsilon_1\varepsilon_3$	3
$34 \cdot 5 =$	$3 \cdot 45 =$	$-\varepsilon_5\varepsilon_6$	$\varepsilon_1\varepsilon_2$	2
$34 \cdot 6 =$	$3 \cdot 46 =$	$-\varepsilon_3\varepsilon_6$	$-\varepsilon_1\varepsilon_4$	1
$34 \cdot 7 =$	$3 \cdot 47 =$	$-\varepsilon_6$	$-\varepsilon_6$	0
$35 \cdot 6 =$	$3 \cdot 56 =$	$-\varepsilon_7$	$-\varepsilon_7$	0
$35 \cdot 7 =$	$3 \cdot 57 =$	$\varepsilon_3\varepsilon_7$	$-\varepsilon_1\varepsilon_5$	1
$36 \cdot 7 =$	$3 \cdot 67 =$	$-\varepsilon_5\varepsilon_7$	$\varepsilon_1\varepsilon_3$	2
$45 \cdot 6 =$	$4 \cdot 56 =$	$\varepsilon_2\varepsilon_3$	$-\varepsilon_6\varepsilon_7$	7
$45 \cdot 7 =$	$4 \cdot 57 =$	$-\varepsilon_2\varepsilon_3$	$-\varepsilon_4\varepsilon_5$	6
$46 \cdot 7 =$	$4 \cdot 67 =$	$-\varepsilon_4\varepsilon_5$	$-\varepsilon_2\varepsilon_3$	5
$56 \cdot 7 =$	$5 \cdot 67 =$	$-\varepsilon_6\varepsilon_7$	$\varepsilon_2\varepsilon_3$	4

Write as before $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +$; then disregarding the lines (such as the first line) which contain the symbol 0, and writing down only the signs as given in the third and fourth columns, these are

ε_6	ε_4	ε_5	$-\varepsilon_6$	$+$	$-\varepsilon_5\varepsilon_7$	$-\varepsilon_6$	$-\varepsilon_4$
ε_7	$-\varepsilon_5$	ε_7	ε_4	$-$	$-\varepsilon_4\varepsilon_6$	ε_7	$-\varepsilon_5$
$-\varepsilon_7$	$-\varepsilon_4$	ε_5	$-\varepsilon_6$	$-\varepsilon_4\varepsilon_7$	$-$	$-\varepsilon_5\varepsilon_7$	$+$
$-\varepsilon_6$	ε_5	$-\varepsilon_4$	$-\varepsilon_7$	ε_4	ε_6	$+$	$-\varepsilon_6\varepsilon_7$
$-\varepsilon_4$	$-\varepsilon_6$	$-\varepsilon_6$	ε_5	$-\varepsilon_5$	ε_7	$-$	$-\varepsilon_4\varepsilon_5$
$-\varepsilon_5$	ε_7	$+$	$-\varepsilon_5\varepsilon_6$	$-\varepsilon_4\varepsilon_6$	$-$	$-\varepsilon_4\varepsilon_5$	$-$
ε_4	ε_7	$-$	$-\varepsilon_4\varepsilon_7$	$-\varepsilon_5\varepsilon_6$	$+$	$-\varepsilon_6\varepsilon_7$	$+$

and we hence see at once that the pairs of signs in the two columns respectively cannot be made identical: to make them so, we should have $\varepsilon_6 = \varepsilon_4$, $\varepsilon_7 = -\varepsilon_5$, $\varepsilon_7 = \varepsilon_4$, that is $\varepsilon_4 = \varepsilon_6 = \varepsilon_7 = -\varepsilon_5$, which is inconsistent with the last equation of the system $-\varepsilon_6\varepsilon_7 = +$. Hence the imaginaries 1, 2, 3, 4, 5, 6, 7, as defined by the original conditions, are not in any case associative.

If we have $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = +$ and also $-\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = \varepsilon_7 = \theta$, that is, if the imaginaries belong to the 8-square formula, then it is at once seen that each pair consists of two opposite signs; that is, for the several triads 123, 145, 167, 246, 257, 347, 356 used for the definition of the imaginaries, the associative property holds good, $12\cdot3 = 1\cdot23$, etc.; but for each of the remaining twenty-eight triads, *the two terms are equal but of opposite signs*, viz. $12\cdot4 = -1\cdot24$, etc.; so that the product 124 of any such three symbols has no determinate meaning.

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